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SOME PROPERTIES OF THREE-TERM FRACTIONAL ORDER SYSTEM

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Abstract

The standard second-order transfer function is of great significance in control theory. Recently, according to the advances in modeling and control by means of fractional derivatives, there has been an increasing need for fractional-order transfer functions generalizing the second-order one. Such transfer functions lead to better results when the plant transfer function consists of fractional powers of the Laplace variable s. Some important features of a three-term fractional system such as stability, settling time, and overshoot are studied, and simulations are presented in this paper.

Mathematics Subject Classification: 26A33; 93C05, 93C55, 93C80

Key Words and Phrases: fractional order system, second order transfer function, fractional order transfer function, settling time, control theory

1. Introduction

The standard second-order transfer function

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \tag{1}$$

is of great significance in control theory. The main reason is that the behavior of many practical systems is dominated by a second order dynamics. Moreover, in many applications such as the model matching problem [4], the model reference adaptive control (MRAC) [2], and the model following [3] there is a need for a reference model which is fulfilled by (1) in the classical case. The step response of (1) is also well studied and formulas for calculating the rise time, settling time, peak time, overshoot etc. can be found in textbooks as [8].

Recently, according to the advances in modeling and control by means of fractional derivatives, the use and studies of fractional-order control systems and fractional-order transfer functions have dramatically increased, see for example [6], [9] and the 10 years' contents of the journal *FCAA*, at *http://www.math.bas.bg/~fcaa/contfc_all_10vol.pdf*.

In this paper, the three-term fractional-order transfer function

$$H(s) = \frac{\omega_n^2}{s^{2/v} + 2\zeta\omega_n s^{1/v} + \omega_n^2}, \tag{2}$$

where $v \in \mathbb{N} \setminus 1$, and $\omega_n \in \mathbb{R}^+$, is introduced as a generalization of (1), and its time domain behavior is studied. The domain of definition for (2) is a Riemann surface with v Riemann sheets, where the origin is a branch point (of order $v-1$) and the branch cut is assumed at $\mathbb{R}^-$. Studying the time-domain behavior of a multi-valued transfer function, as given in (2), is a challenging task. This difficulty is due to the transcendental functions appeared in the inverse Laplace transform. Note that for $v = 1$, (2) is reduced to the conventional case (1). Obviously, (2) has two poles which are distributed on v Riemann sheets.

The major reasons for replacing the standard second-order transfer function (1) with (2) are the following: The curve fitting properties of measured response functions in time and frequency domain improve. Fewer number of parameters are involved for the curve fitting compared to a higher order integer transfer function. Moreover, the model structure (2) is advantageous in that it can be used to model slower systems comparing to (1). It will be shown in Section 4 that the fractional-order transfer function (2) can be used to model the systems, the impulse response of which are as slow as $o\left(t^{-1}\right)$. In the rest of this paper, when it is referred to *transfer function*, *impulse response*, or *step response*, a fractional-order system is meant unless it is stated otherwise.

This paper is arranged as follows. The effect of ζ on stability and location of the poles is studied in Section 2. The impulse and step responses are studied in Section 3. Section 4 contains the main results of this paper. A formula for estimating the settling time is given, and the asymptotic behavior is studied. The effect of parameters on the system response is also discussed. The condition under which the step response is a monotonic function of time is given in Section 5. Finally, Section 6 contains the conclusion.

2. Roots loci and stability condition

It is concluded from [6] that (2) is stable if and only if the roots of the equation

$$w^2 + 2\zeta\omega_n w + \omega_n^2 = 0, \tag{3}$$

lie in the sector

$$|\arg(w)| > \frac{\pi}{2v}. \tag{4}$$

It is resulted from (4) that the stability of (2) depends on v. The roots of (3) are calculated as

$$w_{1,2} = \omega_n \left(-\zeta \pm \sqrt{\zeta^2 - 1} \right),$$

which obviously indicates that stability of (2) does not depend on the value of $\omega_n \in \mathbb{R}^+$. **Figure 1** shows the roots loci of (3) as a function of the varying parameter ζ. The sectors $|\arg(w)| < \frac{\pi}{2v}$ and $\frac{\pi}{2v} < |\arg(w)| < \frac{\pi}{v}$ correspond to the right half plane (RHP) and the left half plane (LHP) of the first Riemann sheet, respectively. In order to find the values of ζ for which the roots of (3) are on the first Riemann sheet we do intersect (3) with the ray $w = re^{i\frac{\pi}{v}}$. It then follows that

$$\left(re^{i\frac{\pi}{v}}\right)^2 + 2\zeta\omega_n \left(re^{i\frac{\pi}{v}}\right) + \omega_n^2 = 0. \tag{5}$$

After some algebra one finds $r = \omega_n$ and

$$\zeta = -\cos\frac{\pi}{v}. \tag{6}$$

Thus, the roots of (3) are on the first Riemann sheet if and only if

$$\zeta < -\cos\frac{\pi}{v}. \tag{7}$$

In the same manner it can be proved that (3) is stable if and only if

$$\zeta > -\cos\frac{\pi}{2v}. \tag{8}$$

Comparing the region of stability of (2) (which is given by (8)) with the region of stability of (1) (that is $\zeta > 0$) it is concluded that (2) has a wider region of stability.

As a result, (2) has a pair of complex conjugate poles on the LHP of the first Riemann sheet if and only if

$$-\cos\frac{\pi}{2v} < \zeta < -\cos\frac{\pi}{v}. \tag{9}$$

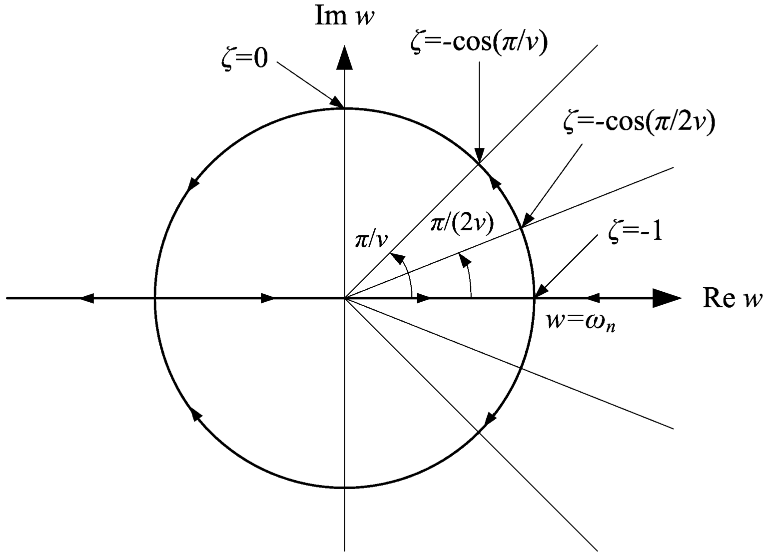

Figure 1: The roots loci of (3) as a function of ζ.

3. Step and impulse responses

In order to calculate the inverse Laplace transform of (2), it should first be expanded as

$$H(s) = \frac{-\frac{\omega_n}{2\sqrt{\zeta^2-1}}}{s^{1/v} + \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}} + \frac{\frac{\omega_n}{2\sqrt{\zeta^2-1}}}{s^{1/v} + \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}}. \tag{10}$$

Then, the inverse Laplace transform of (10) (i.e., the system impulse response) is calculated as (see [7], [9]):

$$\begin{aligned} h(t) = &-\frac{\omega_n}{2\sqrt{\zeta^2-1}} t^{-\frac{v-1}{v}} E_{\frac{1}{v},\frac{1}{v}}\left(-(\zeta\omega_n + \omega_n\sqrt{\zeta^2-1})t^{\frac{1}{v}}\right) \\ &+ \frac{\omega_n}{2\sqrt{\zeta^2-1}} t^{-\frac{v-1}{v}} E_{\frac{1}{v},\frac{1}{v}}\left(-(\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})t^{\frac{1}{v}}\right), \end{aligned} \tag{11}$$

where $E_{\alpha,\beta}(z)$ is the so-called Mittag-Leffler function in two parameters, introduced by Agarwal [1]. A modification of his definition can be found in [5], as

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)} \quad (\alpha > 0, \quad \beta > 0). \tag{12}$$

An excellent MATLAB function for calculating $E_{\alpha,\beta}(z)$ is developed by Podlunby [10] and is freely downloadable from

http://www.mathworks.com/matlabcentral/fileexchange

Although, that function leads to tricky results when t is close to zero, but that is accurate enough for typical applications.

The step response of a system with transfer function (2) is given by

$$S(s) = \frac{\omega_n^2}{s\left(s^{2/v} + 2\zeta\omega_n s^{1/v} + \omega_n^2\right)} \tag{13}$$

$$= \frac{\frac{\zeta-\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}}{s^{1/v} + \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}} - \frac{\frac{\zeta+\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}}{s^{1/v} + \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}} + \frac{1}{s}, \tag{14}$$

in frequency domain which corresponds to the time-domain response

$$\begin{aligned} s(t) = {} & \frac{\zeta-\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}} t^{-\frac{v-1}{v}} E_{\frac{1}{v},\frac{1}{v}}\left(-(\zeta\omega_n + \omega_n\sqrt{\zeta^2-1})t^{\frac{1}{v}}\right) \\ & - \frac{\zeta+\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}} t^{-\frac{v-1}{v}} E_{\frac{1}{v},\frac{1}{v}}\left(-(\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})t^{\frac{1}{v}}\right) + 1. \end{aligned} \tag{15}$$

The system impulse response (11) and step response (15) are in terms of the transcendental functions. This makes it impossible to calculate, for example, the peak time by equating the derivative of (15) to zero. In this paper, (11) and (15) are used to simulate the impulse response and step response, respectively.

4. Settling time

The main purpose of this section is to provide a formula for estimating the settling time of (11), which can be considered as an approximation for the settling time of (15). First note that the impulse response, $h(t)$, can also be calculated by taking the inverse Laplace transform of (2) as follows

$$h(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} H(s)e^{st}ds, \tag{16}$$

where $\sigma \in \mathbb{R}^+$ is bigger than the real part of the rightmost pole of $H(s)$, to ensure causality. In order to evaluate the above integral, consider the contour Γ as shown in **Fig. 2**. First we consider the case in which (2) does not have any pole on the first Riemann sheet. It is concluded from (8) and (7) that this happens if $\zeta > -\cos\frac{\pi}{v}$. In this case we have

$$\oint_\Gamma H(s)e^{st}ds = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} + \int_{C_5} = 0, \tag{17}$$

which, according to (16), yields

$$\int_{C_1} = 2\pi i h(t) = -\left(\int_{C_2} + \int_{C_3} + \int_{C_4} + \int_{C_5}\right), \tag{18}$$

or

$$h(t) = -\frac{1}{2\pi i}\left(\int_{C_2} + \int_{C_3} + \int_{C_4} + \int_{C_5}\right). \tag{19}$$

In the sequel, the above four integrals are evaluated. We parameterize C_2 by considering $s = Re^{i\theta}$, where $R \to \infty$ and $\theta \in [\frac{\pi}{2}, \pi]$. It then follows that

$$\int_{C_2} = \int_{\theta=\frac{\pi}{2}}^{\pi} \frac{\omega_n^2 e^{Re^{i\theta}t} Rie^{i\theta}}{R^{2/v}e^{i2\theta/v} + 2\zeta\omega_n R^{1/v}e^{i\theta/v} + \omega_n^2} d\theta.$$

Using the ML inequality [9] one finds

$$\left|\int_{C_2}\right| \leq ML,$$

where $L = \frac{\pi}{2}$ and

$$M = \max_{\frac{\pi}{2}\leq\theta\leq\pi} \frac{\omega_n^2 e^{Rt\cos\theta} R}{\left|R^{2/v}e^{i2\theta/v} + 2\zeta\omega_n R^{1/v}e^{i\theta/v} + \omega_n^2\right|}.$$

It can easily be verified that the denominator is non-zero for all $\theta \in [\frac{\pi}{2}, \pi]$. Suppose the maximum happens at $\theta = \theta_0 \in [\frac{\pi}{2}, \pi]$. Then clearly $M \to 0$ when $R \to \infty$, which follows that

$$\int_{C_2} = 0. \tag{20}$$

In a similar manner, it can be proved that

$$\int_{C_5} = 0. \tag{21}$$

In order to evaluate the remaining two integrals of (19) we use the parametrization $s = re^{\pm i\pi} \pm i\delta$, $r \in [0, \infty]$ where positive and negative signs correspond to C_3 and C_4, respectively. If $\delta \to 0$ then $s^{2/v} \to r^{2/v}e^{\pm i2\pi/v}$, $e^{st} \to e^{-rt}$, and $ds = -dr$. Thus,

$$\int_{C_3} + \int_{C_4} = \int_0^{\infty} \left(\frac{\omega_n^2 e^{-rt}}{r^{2/v}e^{i2\pi/v} + 2\zeta\omega_n r^{1/v}e^{i\pi/v} + \omega_n^2} - \frac{\omega_n^2 e^{-rt}}{r^{2/v}e^{-i2\pi/v} + 2\zeta\omega_n r^{1/v}e^{-i\pi/v} + \omega_n^2} \right) dr.$$

Short calculations yield

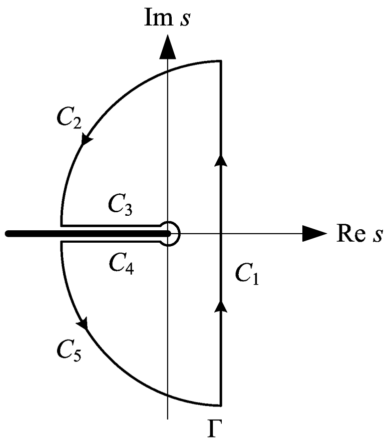

Figure 2: Integration path.

$$\int_{C_3} + \int_{C_4} = 2i\mathrm{Im}\int_0^\infty \frac{\omega_n^2 e^{-rt}}{r^{2/v}e^{i2\pi/v} + 2\zeta\omega_n r^{1/v}e^{i\pi/v} + \omega_n^2}dr, \tag{22}$$

$$= 2i\int_0^\infty \frac{-\omega_n^2\left(r^{2/v}\sin\frac{2\pi}{v} + 2\zeta\omega_n r^{1/v}\sin\frac{\pi}{v}\right)e^{-rt}}{\left|r^{2/v}e^{i2\pi/v} + 2\zeta\omega_n r^{1/v}e^{i\pi/v} + \omega_n^2\right|^2}dr. \tag{23}$$

To give an asymptotic statement, we use Watson's Lemma (see [11]):

Lemma 4.1. *For $\psi(s)$ satisfying $\int_0^\infty |\psi(s)|ds < \infty$ and $\psi(s) \asymp \varrho s^\beta$ in the neighborhood of the origin with (complex) constants ϱ and β, then*

$$\int_0^\infty \psi(s)e^{-sz}ds \asymp \varrho\Gamma(\beta+1)z^{-(\beta+1)}$$

for $|z| \to \infty$ and $|\arg(z)| \le \pi/2 - \delta$, $\delta > 0$.

For $s \mapsto r$ and $z \mapsto t$ we have $\psi(r) \asymp -\frac{2\zeta\sin\frac{\pi}{v}}{\omega_n}r^{1/v}$ in the neighborhood of the origin. Hence, combining (19), (20), (21), and (23) yields

$$h(t) \asymp -\frac{1}{2\pi i}(2i)\frac{-2\zeta\sin\frac{\pi}{v}\Gamma\left(\frac{1}{v}+1\right)}{\omega_n}t^{-\left(\frac{1}{v}+1\right)} \quad \text{for} \quad t \to \infty,$$

or

$$h(t) \asymp \frac{2\zeta\sin\frac{\pi}{v}\Gamma\left(\frac{1+v}{v}\right)}{\pi\omega_n}t^{-\frac{1+v}{v}} \quad \text{for} \quad t \to \infty. \tag{24}$$

It is concluded from the above equation that $h(t)$ is of $o\left(t^{-1}\right)$ for large t when $v \to \infty$.

The settling time, by definition, is the time required for the output to be within the fraction p of the steady state value when the input is subjected to unit step (the commonly used values for p are 0.02 and 0.05). An approximate formula for calculating the settling time, t_s, is obtained by considering the asymptotic behavior of the impulse response. It is concluded from (24) that

$$p = |h(t_s)| \approx \left| \frac{2\zeta \sin\frac{\pi}{v}\Gamma\left(\frac{1+v}{v}\right)}{\pi\omega_n} t_s^{-\frac{1+v}{v}} \right|,$$

or

$$t_s \approx \left[\frac{2|\zeta| \sin\frac{\pi}{v}\Gamma\left(\frac{1+v}{v}\right)}{p\pi\omega_n} \right]^{\frac{v}{v+1}}. \tag{25}$$

This formulae gives a very good insight to the effect of parameters on the characteristic of the step response. For instance, it is concluded from (25) that the settling time is monotonically decreased by increasing ω_n. The settling time is also a monotonically decreasing function of v if $\frac{2|\zeta|}{p\pi\omega_n} > 1$ (which is often the case). However, it is not a good idea to increase the system speed by increasing ω_n because that happens at the cost of larger overshoot (this statement is based on numerous simulations but no proof can be provided at this time). For example, consider a system with $v = 2$, $\zeta = -0.4$, and $\omega_n = 1$. This system can be made faster by increasing either v or ω_n. The step responses are shown in **Figs. 3(a)** and **3(b)** for few values of v and ω_n, respectively. As it is seen, increasing v leads to faster responses with smaller overshoots.

The system impulse response and the corresponding asymptote (24) are shown in **Fig. 4(a)** for when $v = 2$, $\zeta = -0.6$ and $\omega_n = 5$. Considering $p = 0.02$, it is concluded from (25) that $t_s = 2.25$ which perfectly coincides with the one obtained by inspecting the step response plot. Equation (25) does not lead to accurate results when (2) has poles very close the region of instability. The impulse response and the corresponding asymptote for when $v = 2$, $\zeta = -0.7$ and $\omega_n = 5$ are shown in **Fig. 4(b)**. As it is seen, the asymptote dies out much more rapidly than the corresponding impulse response. This is due to the complex conjugate poles that are located very close to the region of instability.

Equation (25) was obtained assuming $\zeta > -\cos\frac{\pi}{v}$. In the sequel, the case $-\cos\frac{\pi}{2v} < \zeta < -\cos\frac{\pi}{v}$ is studied which corresponds to a transfer function with two (stable) poles $s_{1,2} = -\sigma \pm i\Omega$ in Γ. The residue theorem leads to

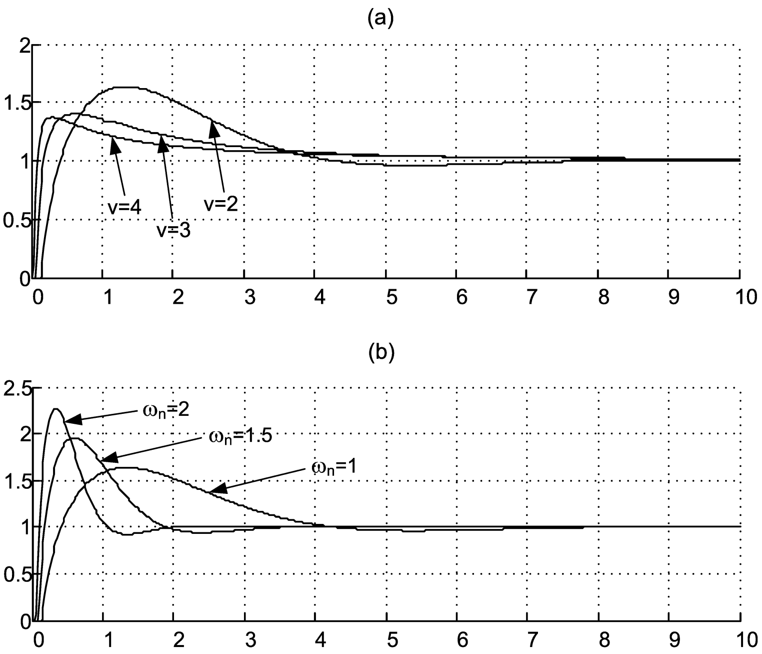

Figure 3: The effect of increasing v and ω_n on the step response.

$$h(t) = \mathrm{Res}_{s_1}\left(H(s)e^{st}\right) + \mathrm{Res}_{s_2}\left(H(s)e^{st}\right) - \frac{1}{2\pi i}\left(\int_{C_2} + \int_{C_3} + \int_{C_4} + \int_{C_5}\right). \tag{26}$$

In the above equation, integrals are calculated as before. With straight calculations one finds

$$K(t) := \mathrm{Res}_{s_1}\left(H(s)e^{st}\right) + \mathrm{Res}_{s_2}\left(H(s)e^{st}\right) = \omega_n^2\left(\frac{e^{s_1 t}}{z'(s_1)} + \frac{e^{s_2 t}}{z'(s_2)}\right), \tag{27}$$

where $z(s) = s^{2/v} + 2\zeta\omega_n s^{1/v} + \omega_n^2$. The function $K(t)$ can be simplified to

$$K(t) = \frac{2\omega_n^2 e^{-\sigma t}}{\mu^2 + \nu^2}(\mu\cos(\Omega t) + \nu\sin(\Omega t)), \tag{28}$$

where $\mu = \Re(z'(s_1))$ and $\nu = \Im(z'(s_1))$. Thus, in this case $K(t)$ represents a damped oscillation. As a result, the right hand side of (26) is dominated by the sum of integrals and consequently, (25) can still be used to approximate the settling time when $-\cos\frac{\pi}{2v} < \zeta < -\cos\frac{\pi}{v}$.

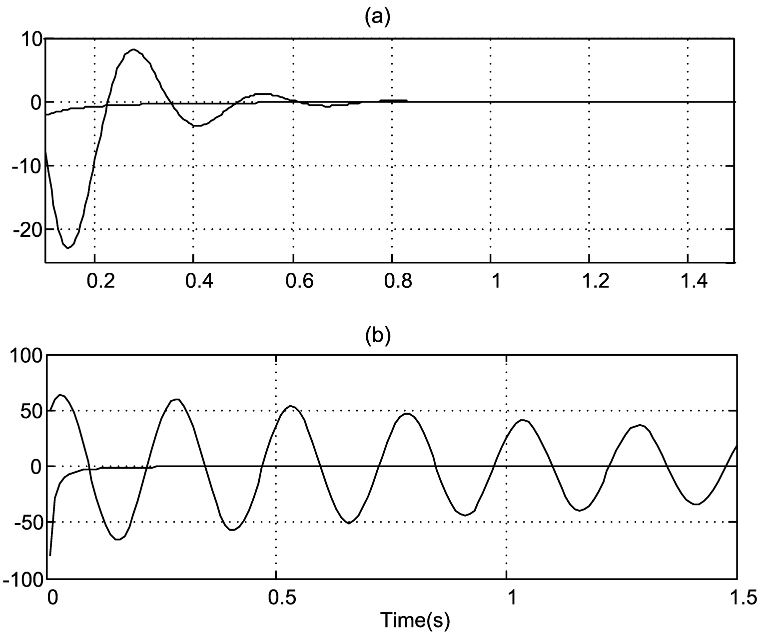

Figure 4: The impulse response and the corresponding asymptote for $v = 2$, $\omega_n = 5$, and (a) $\zeta = -0.6$ (b) $\zeta = -0.7$.

5. Overshoot condition

In this section we are seeking the condition under which the step response of (2) is a monotonic function of time. In fact, the step response $s(t)$ of an asymptotically stable transfer function exhibits overshoot if and only if $s(t) - s(\infty)$ changes sign on $[0, \infty)$. It can easily be verified that

$$\int_0^\infty [s(t) - s(\infty)]e^{-st}dt = \frac{1}{s}H(s) - s(\infty)\frac{1}{s}. \tag{29}$$

The final-value theorem implies that

$$s(\infty) = \lim_{s \to 0} s \times \frac{H(s)}{s} = H(0).$$

Hence, (29) can be written as

$$\int_0^\infty [s(t) - s(\infty)]e^{-st}dt = \frac{H(s) - H(0)}{s}. \tag{30}$$

Let z denote a positive real zero of the asymptotically stable transfer function $H(s) - H(0)$ which is located on the RHP of the first Riemann sheet. Setting $s = z$ in (30) yields

$$\int_0^\infty [s(t) - s(\infty)]e^{-zt}dt = 0.$$

Since e^{-zt} is positive on $[0, \infty)$, it follows that $s(t) - s(\infty)$ must cross zero on $(0, \infty)$. Consequently, $s(t)$ overshoots its steady-state value $s(\infty)$ if $H(s) - H(0)$ has at least one positive real zero on the RHP of the first Riemann sheet. For $H(s)$ given in (2) we have

$$H(s) - H(0) = -\frac{s^{1/v}\left(s^{1/v} + 2\zeta\omega_n\right)}{s^{2/v} + 2\zeta\omega_n s^{1/v} + \omega_n^2},$$

which has one positive real zero at $s = (-2\zeta\omega_n)^v$. Obviously, the zero is on the RHP of the first Riemann sheet if $\zeta < 0$. As a result, the step response of (2) is oscillatory damped if $-\cos\frac{\pi}{2v} < \zeta < 0$.

In what follows, it will be shown that the step response of (2) is a monotonically increasing function of time if $\zeta > 0$. For $\zeta > 0$, the impulse response is given by

$$h(t) = \frac{\omega_n^2}{\pi}\int_0^\infty \frac{\left(r^{2/v}\sin\frac{2\pi}{v} + 2\zeta\omega_n r^{1/v}\sin\frac{\pi}{v}\right)e^{-rt}}{\left|r^{2/v}e^{i2\pi/v} + 2\zeta\omega_n r^{1/v}e^{i\pi/v} + \omega_n^2\right|^2}dr, \tag{31}$$

in which, the integrand is a non-negative function of r for all permissable values of the parameters. Hence, the impulse response does not change sign and the corresponding step response is a monotonically increasing function of time.

Note that, although (2) has neither pole nor zero on the RHP of the first Riemann sheet when $\zeta > -\cos\frac{\pi}{v}$, but the oscillation is caused because of the paths C_3 and C_4 in **Fig. 2**. The problems of estimating the peak time and overshoot remain open.

6. Conclusion

Some important features of the transfer function

$$H(s) = \frac{\omega_n^2}{s^{2/v} + 2\zeta\omega_n s^{1/v} + \omega_n^2} \quad (\omega_n > 0),$$

which is an extension of the classical second-order transfer function to the fractional-order case are studied. It is shown that the above transfer function is stable if and only if $\zeta > -\cos\frac{\pi}{2v}$. Hence, the region of stability is wider comparing to the standard second-order transfer function. It is also proved that the corresponding step response is monotone if and only if $\zeta > 0$. A formula for approximating the settling time with a reasonable accuracy is presented in (25) which is applicable to all stable systems. The effect of the parameters v and ω_n on the settling time is also discussed. In fact, it is shown that the settling time can be decreased by increasing either v or ω_n. The proposed fractional-order transfer function is advantageous in a way that it can be used to model slower systems comparing to the classical second-order transfer function.

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